The Pontrjagin classes in the BSS complex

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We exhibit cocycles in the Bott-Shulman-Stasheff complex which represent the Pontrjagin classes.

1. Introduction

In [1], Dupont introduced a double complex on a simplicial manifold and showed how to use it to construct a homomorphism from $I^*(G)$, the G-invariant polynomial ring over Lie algebra $\mathcal{G}$, to $H^*(BG)$, the cohomology ring of the classifying space BG. This map is called the Bott-Shulman map. There he used the property of $H^*(BG)$ that it is isomorphic to the total cohomology of a complex $\Omega^*(NG(*))$ which is associated to a simplicial manifold $\{NG(*)\}$. The image of the Bott-Shulman map in $\Omega^*(NG(*))$ is called the Bott-Shulman-Stasheff form. In [3], Shulman gave the shape of it.

Theorem 1.1 (Shulman). *In $\Omega^*(NG(*))$, the characteristic classes Φ which are given as the image of Bott-Shulman map is of the following form:*

$$\Phi = \Phi_{q-1} + \Phi_{q-2} + \cdots + \Phi_0$$
$$\Phi_i \in \Omega^{q+i}(NG(q-i)).$$

In [4], the author exhibited a cocycle in $\Omega^*(NG(*))$ which represents the Chern character in the case of $G = GL(n, \mathbb{C})$.

In this paper, as an application of the result in [4], we exhibit cocycles which represent the Pontrjagin classes.

2. The Chern charaters in the BSS complex

In this section we recall the results in [4] that gives cocycles in $\Omega^*(NGL(n, \mathbb{C}))$ which represent the Chern charaters.

we set

$$\varphi_s := h_1 \cdots h_{s-1} dh_s h_s^{-1} \cdots h_1^{-1},$$

$$S_{p-q} = \sum_{\sigma \in \mathfrak{S}_{p-q-1}} \mathrm{sgn}(\sigma) \varphi_{\sigma(1)+1} \cdots \varphi_{\sigma(p-q-1)+1}.$$

Here h_i is the i-th factor of $NG(*)$. Then the following theorem holds.

Theorem 2.1 ([4]). *For* $(1 \le i < j \le p-q+1)$, *we set*:

$$R_{ij} = (\varphi_i + \varphi_{i+1} + \cdots + \varphi_{j-1})^2$$

Then the cocycle in $\Omega^{p+q}(NG(p-q))$ $(0 \le q \le p-1)$ *which represents the p-th Chern character* ch_p *is*

$$\frac{1}{(p-1)!}\left(\frac{1}{2\pi i}\right)^p (-1)^{(p-q)(p-q-1)/2} \times$$
$$\mathrm{tr} \sum (\varphi_1 \wedge H_q(S_{p-q})$$
$$\times \int_{\Delta p-q} \prod_{i<j} (t_{i-1} t_{j-1})^{a_{ij}(Hq(S))} dt_1 \wedge \cdots \wedge dt_{p-q}).$$

Here $H_q(S_{p-q})$ *means the term that* R_{ij} $(1 \le i < j \le p-q+1)$ *are put q -times between* φ_k *and* φ_l *in* S_{p-q} *permitting overlaps;.* $a_{ij}(H_q(S_{p-q}))$ *means the number of* R_{ij} *in it.* $\sum$ *means the sum of all such terms.*

3. The Pontrjagin classes in the BSS complex

In this section we take $G = GL(n, \mathbb{R})$ and exhibit cocycles in $\Omega^*(NG)$ which represent the Pontrjagin classes.

3.1 The first Pontrjagin class in the BSS complex

The invariant polynomials which give the first Pontrjagin class and the second Chern character are the same. Hence we obtain the following theorem.

Theorem 3.1. *The cocycle which represents the first Pontrjagin class in* $\Omega^4(NG)$ *is the sum of the following* $P_{1,3} \in \Omega^3(G)$ *and* $P_{2,2} \in \Omega^2(NG(2))$:

$$P_{1,3} = \left(\frac{1}{2\pi}\right)^2 \frac{-1}{6} \operatorname{tr}(h^{-1}dh)^3,$$

$$P_{2,2} = \left(\frac{1}{2\pi}\right)^2 \frac{1}{2} \operatorname{tr}(dh_1 dh_2 h_2^{-1} h_1^{-1}).$$

3.2 The second Pontrjagin class in the BSS complex

The relation between the invariant polynomials which give the second Pontrjagin class p_2 and the Chern characters is given as: $p_2 = -6\mathrm{ch}_4 + \frac{1}{2}\mathrm{ch}_2^2$. Therefore the following theorems hold true.

Theorem 3.2. *The cochain* $P_{1,7}$ *in* $\Omega^7(G)$ *which corresponds to the second Pontrjagin class is given as follows*:

$$P_{1,7} = (-6)\frac{1}{4!}\left(\frac{1}{2\pi}\right)^2 \frac{1}{35} \operatorname{tr}(h^{-1}dh)^7.$$

Theorem 3.3. *The cochain* $P_{2,6}$ *in* $\Omega^6(NG(2))$ *which corresponds to the second Pontrjagin class is given as follows:*

$$P_{2,6} = \frac{1}{2}P_{1,3}^2 + (-6)\frac{-1}{3!5!}\left(\frac{1}{2\pi}\right)^4$$

$$\begin{aligned}
&\times \operatorname{tr}(3dh_1dh_2h_2^{-1}dh_2h_2^{-1}h_1^{-1}dh_1h_1^{-1}dh_1dh_2h_2^{-1}h_1^{-1}\\
&+3dh_1dh_2h_2^{-1}h_1^{-1}dh_1h_1^{-1}dh_1dh_2h_2^{-1}dh_2h_2^{-1}h_1^{-1}\\
&+6dh_1h_1^{-1}dh_1h_1^{-1}dh_1dh_2h_2^{-1}dh_2h_2^{-1}dh^2h_2^{-1}h_1^{-1}\\
&+6dh_1h_1^{-1}dh_1h_1^{-1}dh_1h_1^{-1}dh_1h_1^{-1}dh_1dh_2h_2^{-1}h_1^{-1}\\
&+6dh_1h_1^{-1}dh_1h_1^{-1}dh_1dh_2h_2^{-1}h_1^{-1}dh_1dh_2h_2^{-1}h_1^{-1}\\
&+6dh_1dh_2h_2^{-1}h_1^{-1}dh_1dh_2h_2^{-1}dh_2h_2^{-1}dh_2h_2^{-1}h_1^{-1}\\
&+6dh_1dh_2h_2^{-1}dh_2h_2^{-1}dh_2h_2^{-1}dh_2h_2^{-1}dh_2h_2^{-1}h_1^{-1}\\
&+2dh_1dh_2h_2^{-1}h_1^{-1}dh_1dh_2h_2^{-1}h_1^{-1}dh_1dh_2h_2^{-1}h_1^{-1}).
\end{aligned}$$

Theorem 3.4. *The cochain* $P_{3,5}$ *in* $\Omega^5(NG(3))$ *which corresponds to the second Pontrjagin class is given as follows*:

$$P_{3,5} = \frac{1}{2}(P_{1,3}P_{2,2} + P_{2,2}P_{1,3}) + (-6)\frac{-1}{3!5!}\left(\frac{1}{2\pi}\right)^4$$

$$\begin{aligned}
&\times \operatorname{tr}(6dh_1h_1^{-1}dh_1h_1^{-1}dh_1dh_2dh_3h_3^{-1}h_2^{-1}h_1^{-1}\\
&-6dh_1h_1^{-1}dh_1h_1^{-1}dh_1h_2dh_3h_3^{-1}h_2^{-1}dh_2h_2^{-1}h_1^{-1}\\
&-6dh_1h_2dh_3h_3^{-1}h_2^{-1}dh_2h_2^{-1}dh_2h_2^{-1}dh_2h_2^{-1}h_1^{-1}\\
&-6dh_1h_2dh_3h_3^{-1}dh_3h_3^{-1}dh_3h_3^{-1}h_2^{-1}dh_2h_2^{-1}h_1^{-1}\\
&+6dh_1dh_2h_2^{-1}h_1^{-1}dh_1dh_2dh_3h_3^{-1}h_2^{-1}h_1^{-1}\\
&-6dh_1dh_2h_2^{-1}h_1^{-1}dh_1h_2dh_3h_3^{-1}h_2^{-1}dh_2h_2^{-1}h_1^{-1}\\
&+6dh_1dh_2dh_3h_3^{-1}h_2^{-1}dh_2dh_3h_3^{-1}h_2^{-1}h_1^{-1}\\
&-6dh_1h_2dh_3h_3^{-1}h_2^{-1}dh_2dh_3h_3^{-1}h_2^{-1}dh_2h_2^{-1}h_1^{-1}\\
&+3dh_1dh_2h_2^{-1}h_1^{-1}dh_1h_1^{-1}dh_1h_2dh_3h_3^{-1}h_2^{-1}h_1^{-1}\\
&-3dh_1h_2dh_3h_3^{-1}h_2^{-1}h_1^{-1}dh_1h_1^{-1}dh_1dh_2h_2^{-1}h_1^{-1}\\
&+3dh_1h_2dh_3h_3^{-1}dh_3h_3^{-1}h_2^{-1}dh_2dh_3h_3^{-1}h_2^{-1}h_1^{-1}\\
&-3dh_1h_2dh_3h_3^{-1}h_2^{-1}dh_2dh_3h_3^{-1}dh_3h_3^{-1}h_2^{-1}h_1^{-1}\\
&+3dh_1dh_2dh_3h_3^{-1}h_2^{-1}h_1^{-1}dh_1h_2dh_3h_3^{-1}h_2^{-1}h_1^{-1}\\
&-3dh_1h_2dh_3h_3^{-1}h_2^{-1}dh_2h_2^{-1}h_1^{-1}\\
&\quad dh_1h_2dh_3h_3^{-1}h_2^{-1}h_1^{-1}\\
&+8dh_1dh_2h_2^{-1}dh_2h_2^{-1}dh_2dh_3h_3^{-1}h_2^{-1}h_1^{-1}\\
&-4dh_1dh_2h_2^{-1}dh_2dh_3h_3^{-1}h_2^{-1}dh_2h_2^{-1}h_1^{-1}\\
&+4dh_1dh_2dh_3h_3^{-1}dh_3h_3^{-1}dh_3h_3^{-1}h_2^{-1}h_1^{-1}\\
&+4dh_1dh_2dh_3h_3^{-1}h_2^{-1}dh_2h_2^{-1}dh_2h_2^{-1}h_1^{-1}\\
&+2dh_1h_1^{-1}dh_1dh_2h_2^{-1}dh_2dh_3h_3^{-1}h_2^{-1}h_1^{-1}\\
&-2dh_1h_2dh_3h_3^{-1}h_2^{-1}dh_2h_2^{-1}dh_2h_2^{-1}h_1^{-1}dh_1h_1^{-1}\\
&+2dh_1dh_2h_2^{-1}dh_2dh_3h_3^{-1}dh_3h_3^{-1}h_2^{-1}h_1^{-1}\\
&-2dh_1h_2^{-1}dh_3h_3^{-1}dh_3h_3^{-1}h_2^{-1}dh_2h_2^{-1}dh_2h_2^{-1}h_1^{-1}\\
&+dh_1h_1^{-1}dh_1dh_2dh_3h_3^{-1}dh_3h_3^{-1}h_2^{-1}h_1^{-1}\\
&-dh_1h_1^{-1}dh_1h_2dh_3h_3^{-1}dh_3h_3^{-1}h_2^{-1}dh_2h_2^{-1}h_1^{-1}).
\end{aligned}$$

Theorem 3.5. *The cochain* $P_{4,4}$ *in* $\Omega^4(NG(4))$ *which corresponds to the second Pontrjagin class is given as follows*:

$$P_{4,4} = \frac{1}{2}P_{2,2}^2 + (-6)\frac{1}{4!3!}\left(\frac{1}{2\pi}\right)^4$$

$$\begin{aligned}
&\times \operatorname{tr}(dh_1dh_2dh_3dh_4h_4^{-1}h_3^{-1}h_2^{-1}h_1^{-1}\\
&-dh_1dh_2dh_3dh_4h_4^{-1}h_3^{-1}dh_3h_3^{-1}h_2^{-1}h_1^{-1}\\
&-dh_1h_2dh_3h_3^{-1}h_2^{-1}dh_2h_3dh_4h_4^{-1}h_3^{-1}h_2^{-1}h_1^{-1}\\
&+dh_1h_2dh_3dh_4h_4^{-1}h_3^{-1}h_2^{-1}dh_2h_2^{-1}h_1^{-1}\\
&-dh_1h_2h_3dh_4h_4^{-1}h_3^{-1}dh_3h_3^{-1}h_2^{-1}dh_2h_2^{-1}h_1^{-1}\\
&+dh_1h_2h_3dh_4h_4^{-1}h_3^{-1}h_2^{-1}dh_2dh_3h_3^{-1}h_2^{-1}h_1^{-1}).
\end{aligned}$$

Errata 1. *There are some typos in [5]. In Theorem 3.2 and Theorem 4.1, the word "to" after "represents" should be omitted. In Theorem 3.2, "folllows" should be replaced with "follows".*

References

[1] J. L. Dupont, Simplicial de Rham cohomology and characteristic classes of flat bundles, Top. Vol 15 (1976), 233-245, Perg Press.

[2] J.L. Dupont, Curvature and Characteristic Classes, Lecture Notes in Math. 640, Springer Verlag, (1978).

[3] H. Shulman, Characteristic Classes and Foliations, Ph.D.Thesis, University of California, Berkeley, 1972.

[4] N. Suzuki, The Chern character in the Simplicial de Rham Complex. Nihonkai Mathematical Journal, Vol.26, No1, pp.1-13, (2015).

[5] N. Suzuki, On some characteristic classes of the universal torus bundle in the BSS complex, 秋田高専研究紀要　第51号, pp.73-75, (平成28年).